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AN ESTIMATE RELATED TO THE STRONG MAXIMUM PRINCIPLE

Haïm Brezis* and Pierre-Louis Lions*

Technical Summary Report #1995
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ABSTRACT

Let A be a second order elliptic operator in divergence form. For every $\lambda > 0$, let u_λ be the unique solution of

$$\begin{cases} Au_\lambda + \lambda u_\lambda = 1 & \text{in } \Omega \\ u_\lambda = 0 & \text{on } \partial\Omega \end{cases}$$

(Ω denotes a domain in $\mathbb{R}^N$ of finite measure). It follows from the maximum principle that $u_\lambda \le \frac{1}{\lambda}$ and from the strong maximum principle that $u_\lambda < \frac{1}{\lambda}$. We provide an explicit bound from below for $\frac{1}{\lambda} - u_\lambda$.

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SIGNIFICANCE AND EXPLANATION

Some of the most useful and important tools in the study of elliptic boundary value problems are maximum principles. So called strong maximum principles provide strict inequalities when appropriate conditions are fulfilled. In this work a strong maximum principle is improved by exhibiting an explicit estimate sharper than what follows from usual arguments.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

AN ESTIMATE RELATED TO THE STRONG MAXIMUM PRINCIPLE

Haïm Brezis* and Pierre-Louis Lions*

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be an open set with finite measure $|\Omega|$. Let A be a second order elliptic operator in divergence form

$$Au = -\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_i b_i \frac{\partial u}{\partial x_i} + cu ,$$

with a_{ij}, b_i, $c \in L^\infty(\Omega)$, $c \geq 0$ a.e. on Ω,

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \qquad \forall\xi \in \mathbb{R}^N, \quad \text{a.e. } x \in \Omega, \quad \nu > 0 .$$

Set

$$M = \left(\sum_i \|b_i\|^2_{L^\infty}\right)^{1/2} .$$

For every $\lambda > 0$, let $u_\lambda \in H^1_0(\Omega) \cap L^\infty(\Omega)$ be the unique solution of

(1) $$Au_\lambda + \lambda u_\lambda = 1 \text{ in } \Omega .$$

(The existence and uniqueness of u_λ follows from Theorem 8.3 in [2] when Ω is bounded; for unbounded domains see [4]. Note that A is not necessarily coercive, and one can not use Lax-Milgram's theorem). It follows from the maximum principle that $u_\lambda \leq \frac{1}{\lambda}$; in fact the strong maximum principle leads to $u_\lambda < \frac{1}{\lambda}$. Our purpose is to provide an explicit bound from below for $|u_\lambda - \frac{1}{\lambda}|$ in terms of λ, ν, M, $|\Omega|$ and N.

We thank Prof. H. Weinberger for drawing our attention to the paper [1].

We start with an easy estimate for u_λ.

Proposition 1. We have

$$u_\lambda(x) \leq \frac{1}{\lambda}\left(1 - e^{-\lambda u_0(x)}\right) \leq \frac{1}{\lambda}\left(1 - e^{-\lambda\|u_0\|_{L^\infty}}\right)$$

where $u_0 \in H^1_0(\Omega) \cap L^\infty(\Omega)$ is the unique solution of

(2) $$Au_0 = 1 .$$

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Proof. Set $v(x) = \frac{1}{\lambda}\left(1 - e^{-\lambda u_0(x)}\right)$; so that $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (since $u_0 \geq 0$ by Theorem 8.1 in [2]). It is easy to verify that

$$Av + \lambda v \geq 1 \quad \text{in} \quad \Omega \; .$$

Thus

$$A(u_\lambda - v) + \lambda(u_\lambda - v) \leq 0 \quad \text{in} \quad \Omega$$

and consequently $u_\lambda \leq v$ in Ω.

We now provide a bound for $\|u_0\|_{L^\infty}$.

Proposition 2. Let u_0 be the solution of (2). We have

$$u_0 \leq \frac{|\Omega|^{2/N}}{\nu} \varphi_N \left(\frac{M|\Omega|^{1/N}}{\nu}\right)$$

where, for each N, φ_N is a continuous function on $\mathbb{R}_+$. In particular when $b_i \equiv 0$ we find

$$u_0 \leq c_N \frac{|\Omega|^{2/N}}{\nu} \, ,$$

when c_N is a constant which depends only on N.

Remark. Combining Propositions 1 and 2 when $A = -\Delta$ we find that

$$\sup_\Omega u_\lambda \leq \frac{1}{\lambda}\left(1 - e^{-\lambda c_N |\Omega|^{2/N}}\right) .$$

A sharper estimate has been obtained in this case by C. Bandle ([1], Theorem 1.1) using symmetrization techniques. For example, if $N = 3$ she finds

$$\sup_\Omega u_\lambda \leq \frac{1}{\lambda}\left(1 - \frac{\lambda^{1/2}R}{\sinh \lambda^{1/2}R}\right)$$

where $\frac{4}{3}\pi R^3 = |\Omega|$; such an estimate is optimal (equality holds when Ω is a ball).

Proof of Proposition 2. We start with the case where $b_i \equiv 0$ and we use a technique of Hartman-Stampacchia [3]. Multiplying (2) by $(u_0 - t)^+$ with $t \geq 0$ leads to

(3) $$\nu \int |\nabla(u_0 - t)^+|^2 \leq \int (u_0 - t)^+ \, .$$

Set

$$\alpha(t) = \text{meas}\{x \in \Omega; \; u_0(x) > t\} \, .$$

In what follows we denote by c various constants depending only on N. We claim that

(4) $$\|(u_0 - t)^+\|_{L^1} \le c\|\nabla(u_0 - t)^+\|_{L^2}\,\alpha(t)^{(N+2)/2N}.$$

Indeed recall an inequality of Nirenberg [5]

(5) $$\|\varphi\|_{L^{N/N-1}} \le c\|\nabla\varphi\|_{L^1} \quad \forall\varphi \in H_0^1(\Omega) \text{ with bounded support.}$$

From (5) we deduce that

(6) $$\|\varphi\|_{L^1} \le \|\varphi\|_{L^{N/N-1}}\,|\mathrm{Supp}\,\varphi|^{1/N} \le c\|\nabla\varphi\|_{L^1}\,|\mathrm{Supp}\,\varphi|^{1/N}$$

$$\le c\|\nabla\varphi\|_{L^2}\,|\mathrm{Supp}\,\varphi|^{(N+2)/2N}$$

Multiplying φ by cut-off functions we see easily that (6) holds for every $\varphi \in H_0^1(\Omega)$. In particular if we choose $\varphi = (u_0 - t)^+$ we deduce from (3) that

$$\nu\|(u_0 - t)^+\|_{L^1} \le c\alpha(t)^{1+2/N}.$$

But

$$\|(u_0 - t)^+\|_{L^1} = \int_t^\infty \alpha(s)ds \equiv \beta(t)$$

and therefore we obtain

(7) $$\beta'\beta^{-N/N+2} + c\nu^{N/N+2} \le 0.$$

Integrating (7) on the interval $(0, \|u_0\|_{L^\infty})$ leads to

$$-\beta(0)^{2/N+2} + c\nu^{N/N+2}\|u_0\|_{L^\infty} \le 0$$

i.e.

$$c\nu^{N/N+2}\|u_0\|_{L^\infty} \le \|u_0\|_{L^1}^{2/N+2}.$$

We conclude using the fact that $\|u_0\|_{L^1} \le |\Omega|\,\|u_0\|_{L^\infty}$.

Proof of Proposition 2 in the general case

Step 1. Suppose first we have an estimate of the form $\|u_0\|_{L^\infty} \leq \varphi(M)$ when $|\Omega| = \nu = 1$. The conclusion of Proposition 2 for the case $|\Omega| \neq 1$, $\nu \neq 1$ follows from a simple homogeneity argument. Therefore we may assume from now on that, for example, $M = 1$ and $\nu = 1$.

Step 2. We shall prove the following:

Lemma 3. Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be the solution of

$$Au + u = 1 . \tag{8}$$

Then $\|u\|_{L^\infty} \leq k$ for some constant $k < 1$ which depends only on $|\Omega|$ and N.

Proof of Lemma 3. Multiplying (8) by $(u - t)^+$, $0 \leq t \leq 1$ we find

$$\int |\nabla(u - t)^+|^2 + \int u(u - t)^+ \leq \int (u - t)^+ + \int |\nabla(u - t)^+|(u - t)^+$$

and consequently

$$\frac{1}{2}\int |\nabla(u - t)^+|^2 \leq (1 - t)\int (u - t)^+ .$$

As in the case where $b_i \equiv 0$ we derive that

$$\|(u - t)^+\|_{L^1} \leq c(1 - t)\alpha(t)^{1+2/N} .$$

But

$$\|(u - t)^+\|_{L^1} = \int_t^1 \alpha(s)ds \equiv \beta(t)$$

and therefore

$$\frac{\beta'}{\beta^{N/N+2}} + \frac{c}{(1 - t)^{N/N+2}} \leq 0 . \tag{9}$$

Integrating (9) on the interval (t,k) where $k = \|u\|_{L^\infty}$ we obtain

$$\beta(t)^{2/N+2} \geq c[(1 - t)^{2/N+2} - (1 - k)^{2/N+2}] . \tag{10}$$

On the other hand if we multiply (8) by $\frac{u}{(1 - u)}$ [(1)] we find

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[(1)] Such a test function was introduced by Trudinger (it is used for example in the proof of Theorem 8.1 in [2]); to make the argument rigorous we should multiply by $\frac{u}{1 + \varepsilon - u}$ and let $\varepsilon \to 0$.

$$\int \frac{|\nabla u|^2}{(1-u)^2} \le \int \frac{|\nabla u|u}{(1-u)} + \int u \le \frac{1}{2}\int \frac{|\nabla u|^2}{(1-u)^2} + \frac{3}{2}|\Omega| \quad (\text{since } u \le 1),$$

and so

$$\int |\nabla \log(1-u)|^2 \le 3|\Omega| .$$

Thus (by (6))

$$\|\log(1-u)\|_{L^1} \le c|\Omega|^{1+1/N},$$

that is

$$\int_0^k \frac{\alpha(s)}{1-s}\, ds \le c|\Omega|^{1+1/N} .$$

Since $\alpha = -\beta'$ we deduce that

$$\int_0^k \frac{\beta(s)}{(1-s)^2}\, ds \le c|\Omega|^{1+1/N} . \tag{11}$$

Set $\theta = \frac{2}{N+2}$; by Hölder's inequality we have

$$\int_0^k \frac{\beta^\theta(s)}{(1-s)^{1+\theta}}\, ds \le \left[\int_0^k \frac{\beta(s)}{(1-s)^2}\, ds\right]^\theta |\log(1-k)|^{1-\theta} \tag{12}$$

$$\le c|\Omega|^{\theta(1+1/N)} |\log(1-k)|^{1-\theta} .$$

Using (10) and (12) we find

$$[|\log(1-k)| - \frac{1}{\theta}(1-(1-k)^\theta)] \le c|\Omega|^{\theta(1+1/N)}|\log(1-k)|^{1-\theta}$$

and finally

$$|\log(1-k)| \le c|\Omega|^{1+1/N} + c .$$

Step 3. We want to estimate $\|u_0\|_{L^\infty}$ where u_0 is the solution of (2). We have

$$Au_0 + u_0 = 1 + u_0$$

and so, by Lemma 3,

$$\|u_0\|_{L^\infty} \le k(1 + \|u_0\|_{L^\infty}) .$$

Hence

$$\|u_0\|_{L^\infty} \le \frac{k}{1-k} \le \frac{1}{1-k} \le c\, e^{c|\Omega|^{1+1/N}} .$$

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(Ω denotes a domain in $\mathbb{R}^N$ of finite measure). It follows from the maximum principle that $u_\lambda \leq \frac{1}{\lambda}$ and from the strong maximum principle that $u_\lambda < \frac{1}{\lambda}$. We provide an explicit bound from below for $\frac{1}{\lambda} - u_\lambda$.